

A note on Erdős-Ko-Rado sets of generators in Hermitian polar spaces

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Abstract

The size of the largest Erdős-Ko-Rado set of generators in the finite classical polar space is known for all polar spaces except for $H(2d-1, q^2)$ when $d \geq 5$ is odd. We improve the known upper bound in this remaining case by using a variant of the famous Hoffman's bound.

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1 Introduction

An Erdős-Ko-Rado set of generators in a finite classical polar space is a set of generators of the polar space that have mutually non-trivial intersection. The largest size of an Erdős-Ko-Rado set of generators in a finite classical polar space was determined in [7] for all finite classical polar spaces except for the hermitian polar space $H(2d-1, q^2)$ of odd rank $d \geq 5$. Here rank means vector space rank and not projective dimension. The best known upper bound for this remaining case was proved in [4]. The idea of the proof was to formulate a linear optimization problem whose solution gives an upper bound. This idea goes back to Delsarte and uses the primitive idempotents of the associations scheme related to set of generators of a polar space, see [1]. In [4] we were however not able to determine the optimal solution of the optimization problem. Using a slightly different approach, the previous bound can be improved as follows.

Theorem 1.1. *If S is an Erdős-Ko-Rado set of generators of $H(2d-1, q^2)$, $d \geq 5$ odd, then*

$$|S| \leq ((q^2 + q + 1)q^{2d-3} + 1) \prod_{\substack{i=1 \\ 2i \neq d \pm 1}}^{d-1} (q^{2i-1} + 1). \quad (1)$$

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Remarks 1. The set consisting of all generators of $H(2d-1, q^2)$, d odd, on a point has size roughly q^{d^2} whereas the bound given in the theorem has size roughly q^{d^2+1} .

2. It can be shown that equality can not occur in the theorem. At the end of Section 2 we sketch a proof of this fact.

3. For small d it can be checked by computer that the given bound is also the solution of the optimization problem mentioned above. I guess this is true for all d , but I did not try to show this.

2 Proof of the theorem

Consider the graph whose vertices are the generators of the hermitian polar space $H(2d-1, q^2)$ of odd rank $d \geq 3$. Let N be the number of generators and number them as G_1, \dots, G_N . For $0 \leq i \leq d$, let A_i be the real symmetric $(N \times N)$ -matrix whose (r, s) -entry is 1, if $G_r \cap G_s$ has rank $d-i$, and 0 otherwise. These real matrices are symmetric and commute pairwise, so they can simultaneously be diagonalized. It is known that there are exactly $d+1$ common eigenspaces V_0, \dots, V_d of these matrices. Also one of the eigenspaces is $\langle j \rangle$ where j is the all one vector of length N . We choose notation so that $V_0 = \langle j \rangle$. If $P_{i,j}$ denotes the eigenvalue of A_j on V_i , then with a suitable ordering of the eigenspaces we have, see [8], Theorem 4.3.6

$$\begin{aligned} P_{i,d} &= (-1)^i q^{(d-i)^2 + i(i-1)}, \\ P_{i,d-2} &= \sum_{u=0}^2 (-1)^{i+u} \begin{bmatrix} d-i \\ 2-u \end{bmatrix} \begin{bmatrix} i \\ u \end{bmatrix} q^{(d-2+u-i)^2 + (i-u)(i-u-1)}. \end{aligned}$$

We want to apply Hoffman's bound (see below) to the generalized adjacency matrix $A := A_d - f A_{d-2}$ where we use for f the value for which the smallest eigenvalue of A is as large as possible. The eigenvalues for A are of course $P_{i,d} - f P_{i,d-2}$, $i = 0, \dots, d$. An investigation shows that the best choice for f is when $P_{d,d} - f P_{d,d-2} = P_{1,d} - f P_{1,d-2}$, which results in the following definition.

$$f := \frac{(q^{d-1} - 1)q^{4(d-2)}}{\begin{bmatrix} d-1 \\ 1 \end{bmatrix} q^{2d-5} - \begin{bmatrix} d-1 \\ 2 \end{bmatrix} + \begin{bmatrix} d \\ 2 \end{bmatrix} q^{d-3}}$$

A direct calculation shows that

$$\begin{aligned} f &= \frac{(q^{2d} - 1)(q^{d-1} - 1)q^{4(d-2)}}{\begin{bmatrix} d \\ 2 \end{bmatrix} (q^{d-2} + 1)(q^{2d-1} - q^{d-2} - q^{d-3} + 1)} \\ &= \frac{(q^2 - 1)(q^4 - 1)(q^{d-1} - 1)q^{4(d-2)}}{(q^{2d-2} - 1)(q^{d-2} + 1)(q^{2d-1} - q^{d-2} - q^{d-3} + 1)} \\ &< q^2 - 1. \end{aligned}$$

Lemma 2.1. *The matrix A has constant row sum*

$$K = q^{d^2} - f \begin{bmatrix} d \\ 2 \end{bmatrix} q^{(d-2)^2} > 0.$$

Proof. The row sum of A is the eigenvalue of A on the eigenspace $\langle j \rangle$. Using $f < q^2 - 1$, it follows that $K > 0$. \square

Lemma 2.2. *The smallest eigenvalue of A is*

$$\lambda := -q^{d(d-1)} + f \begin{bmatrix} d \\ 2 \end{bmatrix} q^{(d-2)(d-3)}$$

and we have $\lambda < -q^{d^2-2d+2}$.

Proof. It follows from the list of eigenvalues that λ is the eigenvalue of $A = A_d - fA_{d-2}$ on the eigenspace V_d . Also, the way we determined f shows that the eigenvalue of A on V_1 is also λ . A straightforward calculation shows that

$$\lambda = -\frac{(q+1)(q^{2d} - q^{2d-3} + q - 1)q^{d^2-d-2}}{(q^{d-2} + 1)(q^{2d-1} - q^{d-2} - q^{d-3} + 1)}$$

and, using this expression, it is easy to see that $\lambda < -q^{d^2-2d+2}$.

The eigenvalue of A on V_0 is K and the previous lemma shows that $K > 0$. For $1 \leq i \leq d-1$, the eigenvalue of A on V_i is $P_{i,d} - fP_{i,d-2}$, and we show in the remaining part of the proof that this eigenvalue is larger than λ .

First consider the case when i is odd. Then in the above formula for $P_{i,d-2}$ as a sum over $u \in \{0, 1, 2\}$, only the term corresponding to $u = 1$ is positive. Hence

$$\begin{aligned} P_{i,d-2} &\leq \begin{bmatrix} d-i \\ 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} q^{(d-1-i)^2 + (i-1)(i-2)} \\ &\leq \frac{q^{(d-i)^2 + i^2 - i + 3}}{(q^2 - 1)^2}. \end{aligned}$$

Using $f < q^2 - 1$, we obtain the following bound for the eigenvalue of A on V_i .

$$\begin{aligned} P_i - fP_{i,d-2} &\geq P_i - f \cdot \frac{q^{(d-i)^2 + i^2 - i + 3}}{(q^2 - 1)^2} \\ &\geq -q^{(d-i)^2 + i^2 - i} - \frac{q^{(d-i)^2 + i^2 - i + 3}}{(q^2 - 1)} \\ &\geq -q^{d^2-2d+2} > \lambda. \end{aligned}$$

Here we have used that $3 \leq i \leq d-2$ (since i is odd).

If i is even, then $P_{i,d} > 0$ and it is not difficult to see that $P_{i,d-2} < 0$. In this case the eigenvalue $P_i - fP_{i,d-2}$ of A on V_i is positive. \square

Lemma 2.3. *If N is the number of generators of $H(2d-1, q^2)$, then*

$$\frac{-\lambda N}{K - \lambda} = ((q^2 + q + 1)q^{2d-3} + 1) \prod_{\substack{i=1 \\ 2i \neq d \pm 1}}^{d-1} (q^{2i-1} + 1).$$

Proof. We denote by f_1 and f_2 the nominator and denominator in the definition of f . We have

$$\begin{aligned} \frac{-\lambda}{K - \lambda} &= \frac{q^{d(d-1)}f_2 - (q^{d-1} - 1)q^{4(d-2)} \begin{bmatrix} d \\ 2 \end{bmatrix} q^{(d-2)(d-3)}}{(q^{d^2} + q^{d(d-1)})f_2 - (q^{d-1} - 1)q^{4(d-2)} \begin{bmatrix} d \\ 2 \end{bmatrix} (q^{(d-1)^2} + q^{(d-2)(d-3)})} \\ &= \frac{q^2 f_2 - (q^{d-1} - 1) \begin{bmatrix} d \\ 2 \end{bmatrix}}{q^2(q^d + 1)f_2 - (q^{d-1} - 1) \begin{bmatrix} d \\ 2 \end{bmatrix} (q^{d-2} + 1)}. \end{aligned}$$

An easy calculation gives

$$f_2(q^{2d} - 1) = \begin{bmatrix} d \\ 2 \end{bmatrix} g$$

where

$$g = (q^4 - 1)q^{2d-5} - (q^{2d-4} - 1) + (q^{2d} - 1)q^{d-3}.$$

Hence

$$\frac{-\lambda}{K - \lambda} = \frac{q^2 g - (q^{d-1} - 1)(q^{2d} - 1)}{q^2(q^d + 1)g - (q^{d-1} - 1)(q^{2d} - 1)(q^{d-2} + 1)}.$$

Thus

$$\begin{aligned} \frac{-\lambda(q^d + 1)}{K - \lambda} &= \frac{q^2 g - (q^{d-1} - 1)(q^d - 1)}{q^2 g - (q^{d-1} - 1)(q^d - 1)(q^{d-2} + 1)} \\ &= 1 + \frac{(q^{d-1} - 1)(q^d - 1)q^{d-2}(1 - q^2)}{q^2 g - (q^{d-1} - 1)(q^d - 1)(q^{d-2} + 1)} \\ &= 1 + \frac{(q^{d-1} - 1)(q^d - 1)q^{d-2}(1 - q^2)}{(q^2 - 1)(q^{2d-1} + 1)(q^{d-2} + 1)} \\ &= 1 - \frac{(q^{d-1} - 1)(q^d - 1)q^{d-2}}{(q^{2d-1} + 1)(q^{d-2} + 1)} \\ &= \frac{(q^2 + q + 1)q^{2d-3} + 1}{(q^{2d-1} + 1)(q^{d-2} + 1)}. \end{aligned}$$

As $N = \prod_{i=1}^d (q^{2i-1} + 1)$, the assertion follows. \square

Let G be a simple and non-empty graph with N vertices v_1, \dots, v_N . A real symmetric $N \times N$ matrix A with diagonal entries zero is called an *extended weight matrix* of G if $A_{rs} \leq 0$ whenever $r \neq s$ and $\{v_r, v_s\}$ is not an edge of the graph G , and if $A_{rs} \neq 0$ for at least one edge $\{v_r, v_s\}$ of the graph. It is called a *K-regular* extended weight matrix, if it has in addition constant row sum K . The following result appeared in various forms in the literature, we present it in the form of Corollary 3.3 in [2] but it already appeared in Lemma 6.1 of [3] when applied to the matrix $A - \lambda I$, and it was also mentioned in [5]. For later application, we sketch the easy proof. Here I stand for the $N \times N$ identity matrix and J for the all-one matrix of the same size.

Result 2.4. *Let Γ be a finite simple and non-empty graph with N vertices and suppose that A is a K -regular generalized weight matrix of G with least eigenvalue λ . Then every independent set S of G satisfies*

$$|S| \cdot (K + |\lambda|) \leq |\lambda| \cdot N.$$

From Result 2.4 applied to $A = A_d - fA_{d-2}$ and from Lemma 2.3, we find that an EKR set S of $H(2d-1, q^2)$ satisfies the inequality (1) of Theorem 1.1. This completes the proof of Theorem 1.1.

We remark that equality in (1) is impossible and sketch a proof. Suppose that $|S|$ satisfies the bound (1) with equality. Then the standard proof of Result 2.4 gives information on the characteristic vector χ of S , in fact, it must lie in the span of the all-one-vector j and the eigenspace of A for the eigenvalue λ . Our arguments show that this eigenspace of A is $V_1 + V_d$, hence $\chi = \frac{|S|}{N}j + v_1 + v_d$ with $v_1 \in V_1$ and $v_d \in V_d$ (here we use $j^\top v_1 = j^\top v_d = 0$, $j^\top j = N$ and $j^\top \chi = |S|$). The vectors v_1 and v_d are also eigenvectors of A_1, \dots, A_d and the eigenvalues are known. Since S is an Erdős-Ko-Rado set, the entries of $A_d \chi$ corresponding to elements of S are zero. This gives a linear equation for the entries a_1 and a_d of v_1 and v_d corresponding to some element of S . A second linear equation comes from $\chi = \frac{|S|}{N}j + v_1 + v_d$, namely $1 = \frac{|S|}{N} + a_1 + a_d$. The two equations are linearly independent, so a_1 and a_d can be calculated and are of course independent of the element of S . With this information the entries of $A_i \chi$ corresponding to elements of S can be calculated for all i , which gives the number of elements of S that meet a given element of S in dimension $d-i$. It turns out that not all these numbers are integers, which is the desired contradiction. The same argument was used in the last section of [6].

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